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Magnons, classical strings and β-deformations

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Abstract: Motivated by the recent work of Hofman and Maldacena [1] we construct a classical string solution on the β-deformed $AdS_5 \times \tilde{S}^5$ background. This string solution is identified with a magnon state of the integrable spin chain description of the $\mathcal{N} = 1$ supersymmetric β-deformed gauge theory. The string solution carries two angular momenta, an infinite J_1 and a finite J_2 which classically can take arbitrary values. This string solution corresponds to the magnon of charge J_2 propagating on an infinite spin chain. We derive an exact dispersion relation for this magnon from string theory.

Keywords: AdS-CFT Correspondence, Duality in Gauge Field Theories, Integrable Field Theories, 1/N Expansion.

Contents

1. Introduction

The AdS/CFT correspondence states the equivalence of string theory on $AdS_5 \times S^5$ to the $\mathcal{N} = 4$ supersymmetric Yang-Mills [2]. One consequence of this duality is that the spectrum of string states must match with the spectrum of operator dimensions in gauge theory. This statement has been tested initially only for supergravity multiplets and their KK descendants. Since quantization of string theory on $AdS_5 \times S^5$ is not fully understood, a more complete verification of the spectral matching has posed substantial difficulties.

A significant new step was made in [3, 4] based on the identification of a particular sector of string states carrying a large angular momentum J with the 'long' gauge theory operators. The authors of [4] have argued that the spectrum of the large-J string states can be computed reliably in the semi-classical approximation.

On the gauge theory side the problem of determining the spectrum corresponds to diagonalizing the dilatation operator. In planar perturbation theory, for small 't Hooft coupling $\lambda = Ng^2 \ll 1$ this problem has an elegant reformulation [5–9] in terms of diagonalizing an integrable spin chain. On the string theory side, the problem simplifies in the opposite limit of $\lambda \gg 1$, where the string sigma-model becomes weakly coupled. It has been argued in [10, 11] that the classical string sigma-model on the $AdS_5 \times S^5$ is also integrable. The appearance of integrability on both sides of the correspondence (albeit in opposite limits) has triggered a lot of interest and offered a hope that the prediction of the matching of the spectra can be tested and verified explicitly.

In the integrable spin chain description of gauge theory, the Yang-Mills composite operators are assembled from specific building blocks which are associated with magnons — the elementary excitations of the spin chain with one flipped spin. The Yang-Mills description of the magnon corresponds to the operator

$$\mathcal{O} \sim \sum_l e^{\mathrm{ipl}} (\cdots ZZZ\, I\, ZZZ \cdots) \tag{1.1}$$

Here the 'impurity' I is inserted at a position l along the chain of Z fields (J of them). In the simplest settings which correspond to the SU(2) sector of the $\mathcal{N} = 4$ gauge theory, the Z field is given by a complex scalar field Φ_1, and the impurity I is given by another complex scalar Φ_2.

Each magnon is characterized by its dispersion relation,

$$E - J = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \tag{1.2}$$

In the limit of a long spin chain, $J \gg \infty$, the magnons are dilute. As a result, the scaling dimensions of Yang-Mills operators can be computed by summing over the dispersion relations (1.2) for each constituent magnon. Equation (1.2) is an exact BPS formula[1] [12] in the $\mathcal{N} = 4$ gauge theory. It can be derived from the supersymmetry algebra [12], or by adopting the calculation in [13].

The AdS/CFT correspondence relates gauge invariant Yang-Mills operators to string states. In the integrable spin chain description, these operators are assembled from magnons. It is natural to ask if the magnon itself has a string description. In a recent paper [1] Hofman and Maldacena have found this description.

Hofman and Maldacena [1] have considered a particular double-scaling $N \to \infty$ limit where

$$J \to \infty\,, \qquad \lambda = \text{fixed}\,, \qquad p = \text{fixed}\,, \qquad E - J = \text{fixed} \tag{1.3}$$

In this limit both, the spin chain of the $\mathcal{N} = 4$ gauge theory, and the classical string on the $AdS_5 \times S^5$, become infinitely long. In this sector the problem of determining the spectra of both theories becomes more tractable. The spectrum of the infinite spin chain can be constructed in terms of asymptotic states. These asymptotic states are made out of elementary excitations of the spin chain – the magnons. They carry a conserved momentum p and have the energy $\epsilon(p) = E - J - 1$ given by eq. (1.2). On the string theory side, the authors of ref. [1] have found a classical solution which precisely corresponds to an elementary magnon of the gauge theory spin chain. The lightcone energy $E - J$ of this classical string coincides with the dispersion relation of the magnon (1.2).

In the subsequent work [14] N. Dorey et al have constructed classical string solutions which correspond to bound states of magnons of the spin chain. There is an exact dispersion relation [14] which holds for the spin chain magnons of charge J_2 and for the semi-classical string with angular momenta J_1 and J_2:

$$E - J_1 \;=\; \sqrt{J_2^2 \;+\; \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \tag{1.4}$$

These results of [1, 14] imply that in the infinite spin chain limit, the $\mathcal{N} = 4$ SYM spectrum represented by elementary and composite magnon asymptotic sates matches precisely with the states in semi-classical string theory on $AdS_5 \times S^5$. Two very recent papers [15, 16] analyze the effects of finite J and of the quantum corrections to the semi-classical string magnons.

[1]More precisely, the dispersion relation is of the form $E - J = \sqrt{1 + f(\lambda) \sin^2 p/2}$. Supersymmetry alone cannot determine the function $f(\lambda)$. However, all known perturbative (up to 3-loops) and the strong-coupling results are consistent with $f(\lambda) = \lambda/\pi^2$.

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The main motivation of this paper is to study what happens with this classical-string/magnon correspondence in theories with less supersymmetry. We will consider an $\mathcal{N}=1$ supersymmetric gauge theory obtained by a marginal β-deformation of the $\mathcal{N}=4$ SYM. The AdS/CFT duality extends to the β-deformed theories where it relates the $\mathcal{N}=1$ β-SYM and the supergravity on the deformed $AdS_5 \times \tilde{S}^5$ background. The gravity dual was found by Lunin and Maldacena in ref. [17]. On the other hand, for real β, the β-SYM theory has an integrable spin chain description [18 − 20, 33] and also the Lax pair exists [22]. Hence, it is interesting to find out if the classical-string/magnon correspondence holds in the β-deformed case.

In [20], it was argued that the two-loop SU(2) spin chain Hamiltonian representing the two-loop dilation operator of the real-β-deformed SYM can be obtained from the 2-loop spin chain Hamiltonian of the undeformed $\mathcal{N}=4$ theory by applying a position-dependent unitary operator $\mathcal{U}$. The effect of this unitary operator is to twist the boundary conditions of the original spin chain. As a result, the asymptotic Bethe ansatz equations for the deformed theory can be obtained from the Bethe ansatz equations of the undeformed theory simply by performing the following substitution

$$p \longrightarrow p - 2\pi\beta \tag{1.5}$$

on the left-hand side of the Bethe ansatz equations. If this construction of the spin chain Hamiltonian of the deformed theory can be extended to higher loops then one can calculate the anomalous dimensions of all single trace operators in the SU(2) sector of the deformed theory by taking the conjectured all-loop Bethe equations for the undeformed theory [7, 8] and modifying it as stated in (1.5). Hence for real-valued deformation parameter $\beta \in \mathbb{R}$, we expect that the dispersion relation (1.4) is modified only through a shift in the momentum p:

$$E - J_1 \;=\; \sqrt{J_2^2 \,+\, \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2} - \pi\beta\right)} \tag{1.6}$$

This modification can also be traced back to a perturbative calculation of anomalous dimensions of operators assembled from (1.1) in the β-deformed theory.

Since in the Hofman-Maldacena limit the magnon momentum p is kept fixed and that for the LM supergravity solution to hold, β must be small, $\beta \ll 1/R \ll 1$, it follows that the dispersion relations for the magnon and multi-magnon states of the β-deformed theory are the same as those in the undeformed case (1.4).

If the Hofman-Maldacena construction does extend to the β-deformed AdS/CFT correspondence, the magnon of the spin chain must correspond to (an open part of) a fundamental string moving on the deformed sphere $\tilde{S}^5$. Since the background itself depends on the deformation parameter $\hat{\gamma}$, one would expect that the relevant classical string solution will carry a lightcone energy $E - J$ that depends nontrivially on the deformation. We will find that this is indeed the case for the β-deformed solution constructed in this paper. From this one might expect that the dispersion relation for magnons of the β-deformed theory would explicitly depend on the deformation parameter $\hat{\gamma} = \beta\sqrt{\lambda}$. If this was true, this would be in contradiction with the integrable description of the β-deformed theory.

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It will turn out that the entire $\hat{\gamma}$-dependence in the dispersion relation for the string is absorbed into the second angular momentum J_2. In the regime where the Lunin-Maldacena supergravity background is reliable ($\beta \ll 1$, $\hat{\gamma}$-fixed) our solution will precisely reproduce the dispersion relation (1.4). The β-deformed classical open string solution corresponds to a magnon of charge J_2 in the spin chain description of the β-deformed gauge theory. This magnon is schematically[2] of the form (1.1) where $Z = \Phi_1$ and the impurity is composite $I \sim (\Phi_2)^{J_2}$.

In this paper we will consider only deformations with β real. In the undeformed $\mathcal{N} = 4$ theory the formula (1.4) is the dispersion relation for states in a short representation. Because it is BPS protected in the $\mathcal{N} = 4$ theory, it is valid for any values of λ. We will re-derive the same formula in the $\mathcal{N} = 1$ theory. In this case we cannot appeal to the BPS properties based on the extended supersymmetry algebra. One particularly pleasing feature of our analysis is that we will be able to derive the full square-root expression in (1.4) from classical string theory. Based on expectations from the integrable spin chain analysis it is likely that the dispersion relation (1.4) is exact and is valid for all values of λ in the β-deformed $\mathcal{N} = 1$ theory.

Recent papers which study perturbative and non-perturbative effects in β-deformed gauge theories include [23 – 32].

2. Classical strings in the β-deformed background

The supergravity background dual to β-deformed gauge theory was constructed by Lunin and Maldacena (LM) [17] by applying a solution generating SL$(3, R)$ transformation to the $AdS_5 \times S^5$ background, or equivalently an STsTS^{-1} transformation. The deformed supergravity solution [17] contains the metric on $AdS_5 \times \tilde{S}^5$

$$ds^2_{\rm str} \;=\; R^2 \left[ds^2_{AdS_5} + \sum_{i=1}^{3} (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + \hat{\gamma}^2\, G\mu_1^2\mu_2^2\mu_3^2 (\sum_{i=1}^{3} d\phi_i)^2 \right] \tag{2.1}$$

where $\tilde{S}^5$ is a β-deformed five-sphere and $\sum_{i=1}^3 \mu_i^2 = 1$. The LM solution also involves the dilaton-axion field τ as well as the RR and NS-NS form fields. In what follows we will require only the expression for the metric (2.1) and the the NS-NS two-form field

$$B_2^{\rm NS} \;=\; \hat{\gamma} R^2 G \, (\mu_1^2\mu_2^2 d\phi_1 d\phi_2 + \mu_2^2\mu_3^2 d\phi_2 d\phi_3 + \mu_3^2\mu_1^2 d\phi_3 d\phi_1) \tag{2.2}$$

Here

$$G^{-1} \;=\; 1 + \hat{\gamma}^2 (\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_1^2\mu_3^2)\,, \qquad R^4 \;:=\; 4\pi N g_{\rm st}\,, \qquad \hat{\gamma} \;:=\; R^2\beta \;=\; \sqrt{\lambda}\beta\,. \tag{2.3}$$

The coordinates (μ_i, ϕ_i) which parameterize the deformed 5-sphere $\tilde{S}^5$ correspond precisely to the three complex scalars Φ_i of the β-deformed gauge theory. This correspondence

[2] More precisely, as explained in [14], the magnon bound-state corresponds to a particular state of the spin chain where the wave-function is strongly peaked on configurations where all flipped spins (i.e. Φ_2's) are nearly adjacent to each other.

is dictated by the three U(1) isometries surviving from the $\mathrm{SU}(4)_R$ symmetry of the $\mathcal{N}=4$ SYM,

$$\mu_1\, e^{i\phi_1} \;=\; \Phi_1 \;=\; \varphi^1 \,+\, i\varphi^2\,, \tag{2.4}$$

$$\mu_2\, e^{i\phi_2} \;=\; \Phi_2 \;=\; \varphi^3 \,+\, i\varphi^4\,, \tag{2.5}$$

$$\mu_3\, e^{i\phi_3} \;=\; \Phi_3 \;=\; \varphi^5 \,+\, i\varphi^6\,, \tag{2.6}$$

Here Φ_i denote complex scalars which are the lowest components of the three chiral superfields of the $\mathcal{N}=1$ supersymmetric β-deformed gauge theory and $\varphi^1,\ldots,\varphi^6$ denote the corresponding six real scalar fields. It will be convenient to parameterize μ_i coordinates via

$$\mu_3 = \sin\theta\sin\alpha\,, \qquad \mu_1 = \sin\theta\cos\alpha\,, \qquad \mu_2 = \cos\theta \tag{2.7}$$

so that $\sum_i d\mu_i^2 = d\theta^2 + \sin^2\theta\, d\alpha^2$.

As mentioned earlier, the LM supergravity background is a reliable approximation to string theory in the regime [17] where $R \gg 1$ and $\beta \ll 1$.

Ultimately we are interested in closed bosonic strings moving on $\mathbb{R}\times\tilde{S}^5$. These closed (folded) strings can be constructed from open strings in the same manner as in [1]. It is an open string with the ends on the equator of the deformed sphere which corresponds to a magnon building block of the gauge theory operators. From now on we concentrate on such open strings.

$\hat{\gamma}=0$**: Hofman-Maldacena solution.** First we briefly recall the classical string solution in the undeformed $\mathcal{N}=4$ theory found by Hofman and Maldacena in [1]. The deformations are switched off by setting $\hat{\gamma}=0$ (which also implies $G=1$ with $B^{\mathrm{NS}}=0$) in eqs. (2.1)-(2.3) above. This solution lives on the $\mathbb{R}\times S^2$ background, where the metric on S^2 is

$$ds^2 \;=\; d\theta^2 \,+\, \sin^2\theta\, d\phi^2 \tag{2.8}$$

We are looking for a solution of equations of motion arising from the Polyakov action in the conformal gauge. It corresponds to a classical open string moving on the infinite worldsheet parameterized by coordinates t and x.

This solution $\theta(x,t)$, $\phi(x,t)$ we are after can be written in the form

$$\theta(x,t) \;=\; \theta(y)\,, \qquad \phi(x,t) \;=\; t \,+\, g(y)\,, \qquad y \;:=\; cx \,-\, dt \tag{2.9}$$

where c and d are positive constants. Functions $\theta(y)$ and $g(y)$ represent a wave localized around $y=0$ and moving with a group velocity $v=d/c\leq 1$. Apart from the y-dependence, the angle ϕ in (2.9) also depends on t linearly. This is interpreted as a string rotating in the azimuthal ϕ-direction and gives rise to a large angular momentum $J=\partial S/\partial\dot{\phi}$. The explicit form of the HM solution reads [1]

$$\cos\theta \;=\; \frac{1}{c}\,\frac{1}{\cosh y}\,, \qquad \tan g \;=\; \frac{1}{d}\tanh y \tag{2.10}$$

where the constants c and d are given by

$$c \;=\; \frac{1}{\cos\theta_0}\,, \qquad d \;=\; \tan\theta_0\,, \qquad c^2-d^2=1 \tag{2.11}$$

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and $\cos\theta_0$ is the maximal value of $\cos\theta$ in (2.10). This solution is characterized by two integrals of motion, the energy E, and the ϕ-angular momentum J. Both E and J are infinite quantities when evaluated on the solution, but the combination $E - J$ is finite,

$$E - J = \frac{\sqrt{\lambda}}{\pi}\cos\theta_0 \tag{2.12}$$

This classical string solution corresponds to an elementary magnon in the integrable spin chain description of the $\mathcal{N} = 4$ gauge theory. Let us choose the azimuthal angle on the S^2 sphere (2.8) to be ϕ_1 for concreteness. This is achieved by setting $\alpha = 0$ on the S^5 so that

$$\mu_1 = \sin\theta\,, \qquad \mu_2 = \cos\theta\,, \qquad \mu_3 = 0 \tag{2.13}$$

In addition we decouple ϕ_2 and ϕ_3 by setting them to zero. The resulting metric on S^2 is given by (2.8) with $\phi = \phi_1$. In the co-moving frame (i.e. in terms of the (t, y) coordinates) the string solution is represented by a time-independent $\theta(y)$ and a time-dependent $\phi_1 = t + g_1(y)$. This corresponds to a string rotating in the azimuthal ϕ_1 direction. In terms of the μ_i coordinates (2.13) the rotation is around μ_1

$$\mu_1\, e^{i\phi_1(t)} = \sin\theta\, e^{i\phi_1(t)}\,, \qquad \mu_2 = \cos\theta\,, \qquad \mu_3 = 0\,, \qquad \theta = \text{const} \tag{2.14}$$

Comparing this with the gauge theory dictionary (2.4)-(2.6), we see that in this case the Hofman-Maldacena string solution corresponds to a magnon (1.1) with $Z = \Phi_1$ and the impurity $I = \Phi_2$. The exact dispersion relation for this magnon is given by

$$E - J_1 = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}} \tag{2.15}$$

which in the large λ limit coincides with the classical string result (2.12) provided that one makes an identification [1]

$$\sin\frac{p}{2} = \cos\theta_0 \tag{2.16}$$

$\hat{\gamma} \neq 0$**: String solution parameterized by two angles.** The main motivation behind this paper is to find what happens to the Hofman-Maldacena solution when one switches on the deformation parameter $\hat{\gamma}$. The most obvious thing seems to be to construct the appropriate solution in terms of $\theta(y)$ and $\phi_1(t, y)$ on the β-deformed background. This will be done in the next sub-section where we will see that the minimal such solution will necessarily involve the third angle, e.g. ϕ_2 and will be forced to propagate on $\tilde{S}^3$ rather than $\tilde{S}^2$.

Before turning to this case we want to comment on a more trivial case of the solution propagating on a 2-sphere which does not involve ϕ_i's, but instead is parameterized by the angles θ and α. For simplicity we set all ϕ_i angles to zero. At this point the deformed $\tilde{S}^5$ sphere collapses to the ordinary S^2 sphere with the metric

$$ds^2 = d\theta^2 + \sin^2\theta\, d\alpha^2 \tag{2.17}$$

There is no deformation left and the resulting classical solution is precisely the undeformed Hofman-Maldacena solution with ϕ replaced by α. The rotation is in the (μ_1, μ_3) plane with

μ_2 being constant, cf. (2.7). This solution describes the magnon of eq. (1.1). However, the Z-fields are not given by any single superfield Φ_i. The rotating field is actually $\varphi^1 + i\varphi^5$. Hence the $\theta - \alpha$ solution corresponds to the magnon (1.1) with $Z = \varphi^1 + i\varphi^5$ and the impurity $I = \Phi_2$. Of course, in the undeformed $\mathcal{N} = 4$ gauge theory, this magnon is equivalent to the magnon of the Hofman-Maldacena solution due to the $\mathrm{SO}(6)_R$ symmetry. The same is true in the β-deformed theory at least at small values of $\beta \ll 1$ relevant for the LM supergravity regime. In this regime the dispersion relation of the magnon is given by (1.2).

$\hat{\gamma} \neq 0$**: String solution parameterized by three angles.** Now we want to study a non-trivial deformation of the Hofman-Maldacena solution in the $\theta - \phi$ sector. It will turn out that this solution is required to live on the $\tilde{S}^3$ sphere. Hence we need to consider a classical string moving on the $\mathbb{R} \times \tilde{S}^3$. To achieve this we set $\alpha = 0$ and use (2.13). The deformed 3-sphere $\tilde{S}^3$ is parameterized by the three angles θ, ϕ_1 and ϕ_2. The non-vanishing components of the metric and the two-form field B^{NS} are given by

$$ds^2 = d\theta^2 + G \sin^2\theta \, d\phi_1^2 + G \cos^2\theta \, d\phi_2^2 \,, \tag{2.18}$$

$$B^{\mathrm{NS}}_{\phi_1\phi_2} = \hat{\gamma} \; G \;\; \sin^2\theta \;\; \cos^2\theta \tag{2.19}$$

Classical equations follow from the Polyakov action

$$S = -\frac{\sqrt{\lambda}}{2} \int \frac{d\tau dx}{2\pi} \sqrt{-\gamma} \left[\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} - \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}\right] \tag{2.20}$$

After fixing the gauge through $\gamma^{\alpha\beta} = \eta^{\alpha\beta} = (-1, 1)$ and plugging in (2.20) the expressions for $G_{\mu\nu}$ and $B^{\mathrm{NS}}_{\mu\nu}$ we have

$$\begin{aligned} S = -\frac{\sqrt{\lambda}}{2} \int \frac{d\tau dx}{2\pi} \, &[-(\partial_\tau t)^2 - (\partial_\tau \theta)^2 + (\partial_x \theta)^2 + G\cos^2\theta((\partial_x \phi_2)^2 - (\partial_\tau \phi_2)^2) \\ &+ G\sin^2\theta((\partial_x\phi_1)^2 - (\partial_\tau\phi_1)^2) - 2\hat{\gamma} G \sin^2\theta\cos^2\theta(\partial_\tau\phi_2\partial_x\phi_1 - \partial_x\phi_2\partial_\tau\phi_1)]. \end{aligned} \tag{2.21}$$

Equations of motion for t, θ, ϕ_1 and ϕ_2 follow from this action.[3] We choose the conformal gauge $t = \tau$ and look for the classical solution of the form

$$\theta(x,t) \,=\, \theta(y)\,, \qquad \phi_1(x,t) \,=\, t \,+\, g_1(y)\,, \qquad \phi_2(x,t) \,=\, g_2(y) \tag{2.22}$$

Here $y = cx - dt$ is the same as in the undeformed case. The constants c and d are real-valued and positive. The main difference of the ansatz (2.22) with that in the undeformed case (2.9) is the appearance of the third angle $\phi_2 = g_2(y)$. It follows from the equations of motion (and in particular from the contributions of the B^{NS} form) that ϕ_2 can never be decoupled on the deformed sphere with $\hat{\gamma}^2 > 0$. This implies that in the (θ, ϕ_i) sector any β-deformation of the S^2-solution of Hofman and Maldacena will necessarily live on the 3-sphere $\tilde{S}^3$. Generalizations to motion on higher spheres is straightforward.

[3] We have also solved classical equations which follow from the Nambu-Goto action. In this way we found the same solutions and the same expression for the energy as the ones written down below. This agreement also guarantees that our solutions satisfy the Virasoro constraints.

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The classical equations for our ansatz take the form

$$(d^2-c^2)\partial_y^2\theta = \tfrac{1}{2}\partial_\theta(G\sin^2\theta)\left(1-2d\partial_y g_1+(d^2-c^2)(\partial_y g_1)^2\right)+\tfrac{1}{2}\partial_\theta(G\cos^2\theta)(d^2-c^2)(\partial_y g_2)^2$$
$$+\partial_\theta(\hat\gamma G\sin^2\theta\cos^2\theta)c\partial_y g_2 \tag{2.23}$$

$$(d^2-c^2)\,\partial_y(G\,\sin^2\theta\,\partial_y g_1) - d\,\partial_y(G\,\sin^2\theta) = 0 \tag{2.24}$$
$$(d^2-c^2)\,\partial_y(G\,\cos^2\theta\,\partial_y g_2) - c\,\partial_y(\hat\gamma G\,\sin^2\theta\,\cos^2\theta) = 0 \tag{2.25}$$

These equations can be simplified as follows

$$\partial_y\, g_1 = -d\left(1-\frac{1}{G\,\sin^2\theta}\right) \tag{2.26}$$
$$\partial_y\, g_2 = -\hat\gamma\, c\,\sin^2\theta \tag{2.27}$$
$$(\partial_y\,\theta)^2 = c^2\,\cos^2\theta + d^2\left(1-\frac{1}{G\,\sin^2\theta}\right) \tag{2.28}$$

Here we have imposed $c^2-d^2=1$ which guarantees that the group velocity $v\equiv d/c\leq 1$. We also have applied boundary conditions that as $y\to\pm\infty$ the angle $\theta\to\pi/2$. At the same time the derivatives $\partial_y\,\theta$ and $\partial_y\,g_1$ vanish in this limit. It is easy to see that the derivative of the third angle, ϕ_2, cannot be vanishing at infinity, $\partial_y\,g_2\to -\hat\gamma c\neq 0$.

Substituting the expression (2.3) for G into the equation for θ we get

$$\partial_y\,\theta = \cos\theta\sqrt{c^2 - d^2\,\frac{1+\hat\gamma^2\sin^2\theta}{\sin^2\theta}} \tag{2.29}$$

This equation can be integrated and admits an analytic solution:

$$\cos\theta = \sqrt{\frac{1-\hat\gamma^2 d^2}{c^2-\hat\gamma^2 d^2}}\,\frac{1}{\cosh\left(\sqrt{1-\hat\gamma^2 d^2}\;y\right)} \equiv \cos\theta_0\,\frac{1}{\cosh\left(\sqrt{1-\hat\gamma^2 d^2}\;y\right)} \tag{2.30}$$

This expression is reminiscent of the undeformed solution in (2.10). Solutions of the two remaining equations (2.26), (2.27) can be found straightforwardly from the expression for $\cos\theta$ in (2.30).

We now proceed to evaluate the conserved charges corresponding to the t, ϕ_1 and ϕ_2 isometries of the background. These are the energy E and the two angular momenta J_1 and J_2. They are given by

$$E = \int_{-\infty}^{\infty} dx\,\delta S/\delta\dot t = \frac{\sqrt\lambda}{2\pi}\int_{-\infty}^{\infty} dx\,, \tag{2.31}$$
$$J_1 = \int_{-\infty}^{\infty} dx\,\delta S/\delta\dot\phi_1 = \frac{\sqrt\lambda}{2\pi}\int_{-\infty}^{\infty} dx\;G\sin^2\theta(\partial_t\phi_1-\hat\gamma\cos^2\theta\partial_x\phi_2)\,, \tag{2.32}$$
$$J_2 = \int_{-\infty}^{\infty} dx\,\delta S/\delta\dot\phi_2 = \frac{\sqrt\lambda}{2\pi}\int_{-\infty}^{\infty} dx\;G\cos^2\theta(\partial_t\phi_2+\hat\gamma\sin^2\theta\partial_x\phi_1) \tag{2.33}$$

With these expressions we can now derive the expression for the quantity $E - J_1$, on our solution. We substitute into (2.32) the solutions (2.22)-(2.27), (2.30) and after a little algebra we find a simple result

$$E - J_1 \;=\; \frac{\sqrt{\lambda}}{2\pi}\, c^2 \int_{-\infty}^{\infty} dx\, \cos^2\theta \;=\; \frac{\sqrt{\lambda}}{2\pi}\, c \int_{-\infty}^{\infty} dy\, \cos^2\theta \tag{2.34}$$

This expression can be rewritten entirely in terms of the angle θ,

$$E - J_1 \;=\; \frac{\sqrt{\lambda}}{\pi}\, c \int_{\theta_0}^{\pi/2} d\theta\, \cos^2\theta\, (\partial_y\theta)^{-1} \tag{2.35}$$

Using (2.29) the integral in (2.35) can be performed immediately with the result being

$$E - J_1 \;=\; \frac{\sqrt{\lambda}}{\pi} \sqrt{1+\hat{\gamma}^2\sin^2\theta_0}\;\cos\theta_0\,, \qquad 0 \le \theta_0 \le \pi/2 \tag{2.36}$$

The expression for the second angular momentum J_2 can be also simplified in a similar fashion giving us the simple result

$$J_2 \;=\; \frac{\sqrt{\lambda}}{\pi}\,\hat{\gamma}\, d \int_{-\infty}^{\infty} dy\, \cos^2\theta \;=\; \hat{\gamma}\,\frac{d}{c}\,(E - J_1)\;. \tag{2.37}$$

For ease of comparing different formulae, it will be useful to express all the answers in terms of θ_0. It follows from (2.30) that $\sin^2\theta_0 = d^2/(c^2 - \hat{\gamma}^2 d^2)$. This gives the relation for $d/c = \sin\theta_0/\sqrt{1+\hat{\gamma}^2\sin^2\theta_0}$. This implies that J_2 on our classical string solution takes the form

$$J_2 = \frac{\sqrt{\lambda}}{\pi}\,\hat{\gamma}\;\sin\theta_0\;\cos\theta_0. \tag{2.38}$$

This is a remarkable and somewhat surprising result: our embedding of the Hofman-Maldacena giant magnon solution to the β-deformed theory has resulted in the appearance of the second angular momentum J_2. We recall that J_2 was identically zero on the original Hofman-Maldacena solution in the undeformed theory.[4] In the deformed case $\hat{\gamma} = \text{fixed} \neq 0$ and thus our solution necessarily acquires the second angular momentum J_2. The value of J_2 in (2.38) is proportional to the parameter $\sqrt{\lambda}\hat{\gamma}$ which can take any value, large or small. Furthermore, J_2 in (2.38) depends on the value of θ_0 which labels different solutions within our ansatz. We will clarify the nature of J_2 and the interpretation of the classical string solution as magnon excitations on the spin chain in the next section.

Since our solution has two angular momenta, J_1 and J_2 we can ask if its dispersion relation is reminiscent of (1.4). Remarkably, the dispersion relation is precisely of the square-root form required in (1.4). Using eqs. (2.36) and (2.38), we find that on our solution

$$(E - J_1)^2 - J_2^2 \;=\; \frac{\lambda}{\pi^2}\cos^2\theta_0 \tag{2.39}$$

[4]This of course is not inconsistent with the fact that the Hofman-Maldacena solution corresponds to an elementary magnon of magnon-charge $J_2 = 1$. The vanishing J_2 of the classical string only implies that J_2 is zero at order-$\sqrt{\lambda}$, where $\lambda \gg 1$ to justify the semiclassical analysis.

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Since we want to interpret our solution as a magnon, we will define the magnon momentum p in terms of the parameter θ_0 similarly to the undeformed case (2.16) via

$$\sin\left(\frac{p}{2} - \pi\beta\right) = \cos\theta_0 \tag{2.40}$$

Then the dispersion relation is

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2} - \pi\beta\right)} \tag{2.41}$$

where p is the momentum carried by the magnon. We note that the dispersion relation above depends periodically on p as $p \to p + 2\pi$, and on β as $\beta \to \beta + 1$ as required. In the regime $\lambda \gg 1$, $E \sim J_1 \to \infty$ and J_2 arbitrary[5] this classical string result is completely reliable. Essentially, one expects that the only effect of quantum corrections in this regime is the fact that the angular momenta are quantized, see also [16].

3. Interpretation in terms of magnons

The string solution constructed in section 2 is a generalisation of the Hofman-Maldacena solution to the β-deformed background. We can think of it as the β- or $\hat{\gamma}$-deformation of the Hofman-Maldacena classical string. In the limit where the deformation parameter goes to zero, $\hat{\gamma} \to 0$, our solution collapses to the original Hofman-Maldacena solution, as expected, and can be seen from eqs. (2.36),(2.38). The Hofman-Maldacena solution of the undeformed theory carried a single spin, J_1, and was identified in [1] with an elementary magnon excitation of the spin chain. We have already noted that the deformed solution, in addition to J_1, carries also a non-zero value of the second spin, J_2 given by (2.38). As such, this deformed solution should describe a magnon excitation of magnon-charge $J_2 \propto \sqrt{\lambda}\hat{\gamma}$ in the β-deformed theory. What happens is that when we start with the elementary magnon described by the Hofman-Maldacena solution in the undeformed theory and then turn on the deformation $\hat{\gamma}$ of the background, this induces the charge J_2 and the resulting string configuration corresponds to J_2-boundstate of elementary magnons. If this is the case, then we need to explain how to construct the elementary magnon in terms of a classical string in the deformed theory. This will become clear momentarily.

Note that the solution in the deformed theory we have studied so far, corresponds to a magnon of charge J_2 with a *fixed* momentum p, such that $J_2 = (\sqrt{\lambda}\hat{\gamma}/2\pi)\sin(p - 2\pi\beta)$. This is simply a reflection of the fact that our solution describes a minimal deformation of the Hofman-Maldacena solution, both solutions depend on a *single* free parameter, θ_0 (or c, d with $c^2 - d^2 = 1$, or p). In order to describe magnons with two independent parameters, J_2 and p, one needs to extend the ansatz (2.22) to include dependence on one additional parameter. This is easily achieved by looking for a string solution with two spins in the form [15, 16]

$$\theta(x,t) = \theta(y)\,, \qquad \phi_1(x,t) = t + g_1(y)\,, \qquad \phi_2(x,t) = \nu t + g_2(y) \tag{3.1}$$

[5]Dictated by (2.38).

Here $y = cx - dt$ is the same as before with $c^2 - d^2 = 1$. The new parameter is ν appearing in the equation for ϕ_2. If one sets $\nu = 0$, eqs. (3.1) we are back to the original ansatz (2.22) of section 2.

This ansatz was used earlier in [15] to obtain string solutions with two spins J_1, J_2 in the undeformed theory. In the paper [33] which has appeared after the first version of the present paper, the ansatz (3.1) was used to study corresponding solutions in the β-deformed theory. The integrals of motion of the ν-extended string solution in the deformed theory were calculated in [33]:

$$J_2 \;=\; \frac{\nu c + \hat{\gamma} d}{c}\,(E - J_1)\,, \qquad E - J_1 \;=\; \frac{\sqrt{\lambda}}{\pi}\frac{c\,\cos^2\theta_0}{\sqrt{1-(\nu c+\hat{\gamma}d)^2}} \tag{3.2}$$

where the angular parameter θ_0 is defined in terms of c, d and ν parameters of the solution via

$$\cos^2\theta_0 \;=\; \frac{1-(\nu c+\hat{\gamma}d)^2}{c^2-(\nu c+\hat{\gamma}d)^2} \tag{3.3}$$

These equations generalise expressions in (2.37),(2.36),(2.30) respectively. In particular, it follows that expressions for J_1 and $\cos^2\theta_0$ can be obtained from the results of section 2 by a shift $\hat{\gamma}d \longrightarrow \nu c + \hat{\gamma}d$. In what follows it will be convenient to denote this universal combination as

$$\Gamma \;:=\; \nu c + \hat{\gamma} d \tag{3.4}$$

We will now demonstrate that the two free parameters of the ansatz (3.1) can be chosen in such a way that the resulting giant magnon solution carries a fixed value of the magnon charge J_2 for any value of the magnon momentum p. In other words, one can characterise the giant magnon by two independent arbitrary constant values of p and J_2. In particular, the value of J_2 can be chosen to be one (or more precisely zero in the leading order in $\sqrt{\lambda}$) to describe the elementary magnon, or different from one to describe a magnon boundstate.

The magnon momentum p is determined via (2.40) in terms of $\cos\theta_0$. We now fix the value of p (or equivalently of $\cos\theta_0$) and of the spin J_2 and solve for the free parameters of the ansatz in terms of these values. From eqs. (3.2) we determine the constant Γ in terms of J_2 and p (i.e. θ_0) as

$$\Gamma^2 \;=\; \frac{J_2^2}{J_2^2 + \frac{\lambda}{\pi^2}\cos^4\theta_0}\;. \tag{3.5}$$

Then all the parameters of the ansatz: c, d and ν, are determined through Γ via eqs. (3.3), (3.4) as

$$c^2 \;=\; \Gamma^2 + \frac{1-\Gamma^2}{\cos^2\theta_0}\,, \quad d^2 \;=\; c^2 - 1\,, \quad \nu \;=\; \frac{\Gamma - \hat{\gamma}d}{c}\;. \tag{3.6}$$

Thus we have uniquely fixed the two independent parameters of the classical string in terms of the magnon charge J_2 and momentum p. The dispersion relation still takes the required form (2.41). For $J_2 = 1$ the solution describes the elementary magnon, and for $J_2 > 1$, a magnon boundstate[6] in the β-deformed theory. When set the deformation parameter to

[6] In full quantum theory all angular momenta are quantized. Hence when quantum corrections are included, the classical result for J_2 will have to take integer values.

zero, $\hat{\gamma} \to 0$, the first equation in (3.2) gives precisely the value of $J_2 \propto \nu$ in the undeformed theory as expected [15, 33].

To summarise, in this paper we have shown that the Hofman-Maldacena construction [1] of magnons in terms of classical string solutions can be successfully generalised to β-deformed theories. This generalisation always results in a magnon solution which carries a second orbital momentum J_2. The solution satisfies the exact square-root-type dispersion relation (2.41).

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